Note: If A is a square matrix then T is one-to-one if and only if it's onto

lecture 12

6_ Eigen values and Eigen Vectors

Let A be an $m \times n$ matrix. The scalar λ is called an eigen value of A if there is a non zero vector $\underline{x}$ such that $A\underline{x} = \lambda \underline{x}$

eigen vector

eigen value

Ex $A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$, $\underline{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$A\underline{x}_1 = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad A\underline{x}_1 = 2\,\underline{x}_1$$

eigen value = 2

$\underline{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$A\underline{x}_2 = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad A\underline{x}_2 = -1\,\underline{x}_2$$

eigen value = -1

Ex $A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $\underline{x}_1 \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$

$$A\underline{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \qquad A\underline{x}_1 = 0\,\underline{x}_1$$

eigen value = 0

How to find the eigen values of the eigen vectors?

$A\underline{x} = \lambda x$

$= \lambda I \underline{x}$ $\qquad \lambda I \underline{x} - A\underline{x} = 0$

$(\lambda I - A)\underline{x} = 0$

Det $(\lambda I - A)$ must be zero so that $\underline{x}$ is not trivial and λ has many values

Ex. Find the eigen values and corresponding eigen vectors of $A = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}$

det $(\lambda I - A) = 0$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{pmatrix} \rightarrow \text{characteristic equation}$$

det: $\lambda^2 + 3\lambda + 2 = 0$ $\qquad (\lambda+1)(\lambda+2) = 0$

$\boxed{\lambda = -1, -2} \rightarrow$ eigen values

for $\lambda = -1$

$$\begin{pmatrix} -3 & 12 \\ -1 & 4 \end{pmatrix} \underline{x} = 0 \qquad \begin{pmatrix} -1 & 4 \\ -3 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -1 & 4 \\ -3 & 12 \end{pmatrix} \xrightarrow{R_2 - 3R_1} \begin{pmatrix} -1 & 4 \\ 0 & 0 \end{pmatrix}$$

let $x_2 = t$

$-x_1 + 4x_2 = 0 \qquad x_1 = 4t$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 4 \\ 1 \end{pmatrix} \rightarrow \text{eigen vector}$$

for $\lambda = -2$

$$\begin{pmatrix} -4 & 12 \\ -1 & 3 \end{pmatrix} \xrightarrow[\text{form}]{\text{echelon}} \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix}$$

let $x_2 = s$

$x_1 = 3s$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Ex $A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ characteristic equation $\det(\lambda I - A) = 0$

$$\det \begin{pmatrix} \lambda-1 & -3 & 0 \\ -3 & \lambda-1 & 0 \\ 0 & 0 & \lambda+2 \end{pmatrix} = \lambda+2 \begin{pmatrix} \lambda-1 & -3 \\ -3 & \lambda-1 \end{pmatrix}$$

* to find characteristic equation easily ~~mat~~ substract entries in main diagonal from λ and change the signs of the remaining entries

$\det = (\lambda+2)(\lambda^2 - 2\lambda - 8) = (\lambda+2)(\lambda+2)(\lambda-4) = 0 \qquad \lambda = -2, 4$

for $\lambda = -2$

$$\begin{pmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow[\text{echelon form}]{R_2 - R_3} \begin{pmatrix} -3 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

let $x_2 = t \quad x_3 = s$

$\Rightarrow -3x_1 - 3x_2 = 0$

$x_1 = -t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

eigen vectors

for $\lambda = 4$

$$\begin{pmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \xrightarrow{\text{echelon form}} \begin{pmatrix} 3 & -3 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

let $x_2 = t \quad 6x_3 = 0 \quad x_3 = 0$

$x_1 = t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Special case (triangular matrix)

Theorem: If A is an $n \times n$ ~~mat~~ triangular matrix, then its eigen values are the entries on its main diagonal

eg $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\lambda = 2, 2, 2$

Diagonalisation

Similar Matrices: Two $m \times n$ matrices A and B are called Similar if $B = P^{-1} A P$ holds for some invertible matrix P

- If A and B are similar, they must have
1- Same determinant
2- Same rank
3- Same eigen values
4- Same trace

* Trace is the sum of all entries in main diagonal.

Diagonalizable matrix: An $m \times n$ matrix A is diagonalizable if A is similar to a diagonal matrix

$$P^{-1} A P = \text{diagonal matrix}$$

Steps for diagonalising an $m \times n$ matrix

1- Find n linearly independent eigen vectors $P_1, P_2, \dots, P_n$ for the matrix A with corresponding eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$. If n linearly independent eigen vectors do not exist then A is not diagonalizable
2- let P be the $m \times n$ matrix whose columns consists of these eigen vectors
3- The diagonal matrix $D = P^{-1} A P$ will have the eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$ on its main diagonal (and zeros elsewhere)

eg. $A = \begin{pmatrix} 3 & 2 & 1 \\ 5 & 8 & 0 \\ 0 & 2 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow$ eigenvectors

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

6. Show that matrix A is diagonalizable then find a matrix P such that $P^{-1}AP$ is diagonal

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{pmatrix}$$

1. characteristic equation

$$\det \begin{pmatrix} \lambda-1 & 1 & 1 \\ -1 & \lambda-3 & -1 \\ 3 & -1 & \lambda+1 \end{pmatrix} = 0 \implies \lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0$$

$$(\lambda-3)(\lambda-2)(\lambda+2) = 0$$

$$\lambda = 3,\ 2,\ -2$$

Since the roots are different $\rightarrow$ A is diagonalizable

$\lambda = 3$

$$\begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

echelon $\Downarrow$ form

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{let } x_3 = t \qquad x_1 = -t \quad x_2 = t$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = 2$

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = -2$

$$\begin{pmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{pmatrix} = \frac{t}{4} \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

2- $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 4 \end{pmatrix}$ $\quad (P|I) \Rightarrow (I|P^{-1})$ OR $P^{-1} = \frac{1}{|P|} \operatorname{adj}(P)$

$$P^{-1} = \begin{pmatrix} 0 & -1 & -1 \\ \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{1}{5} & \frac{1}{5} & 1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Ex. Diagonalize the matrix $A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$

characteristic equation

$$\det \begin{pmatrix} \lambda-5 & -8 & -16 \\ -4 & \lambda-1 & -8 \\ 4 & 4 & \lambda+11 \end{pmatrix} = 0 \qquad (\lambda+3)^2(\lambda-1) = 0 \qquad \lambda = -3, -3, 1$$

$\lambda = -3$

$$\begin{pmatrix} -8 & -8 & -16 \\ -4 & -4 & -8 \\ 4 & 4 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = 1$

$$\begin{pmatrix} -4 & -8 & -16 \\ -4 & 0 & -8 \\ 4 & 4 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 3 \\ -1 & -1 & -2 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

THE END!